Entanglement and interference resources in quantum computation and communication

Dan Stahlke

July 11, 2014

"(Entanglement + interference) resources in quantum (computation + communication)"

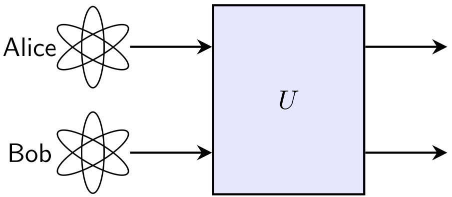
	Entanglement	Interference
Computation	Chapter 2	Chapter 3
Communication	Chapter 2,4,5	Chapter 3

Chapter 2

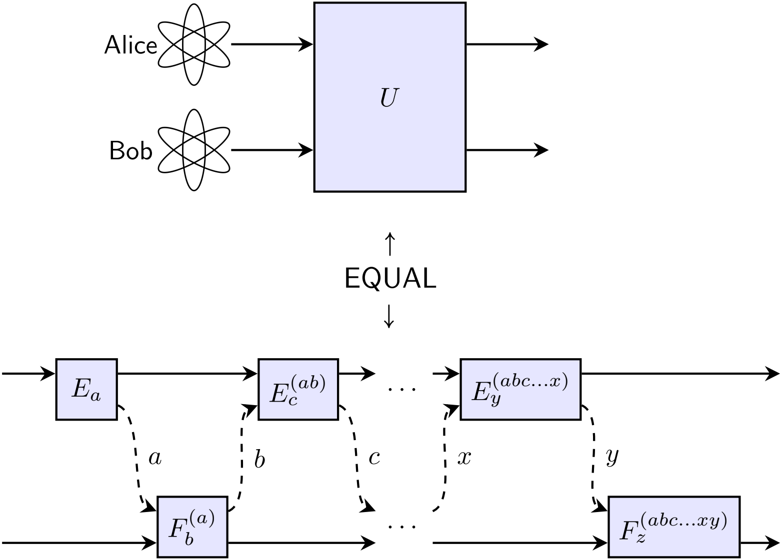
Entanglement requirements for implementing bipartite unitary operations

Dan Stahlke and Robert B. Griffiths,
Phys. Rev. A **84**, 032316 (2011).

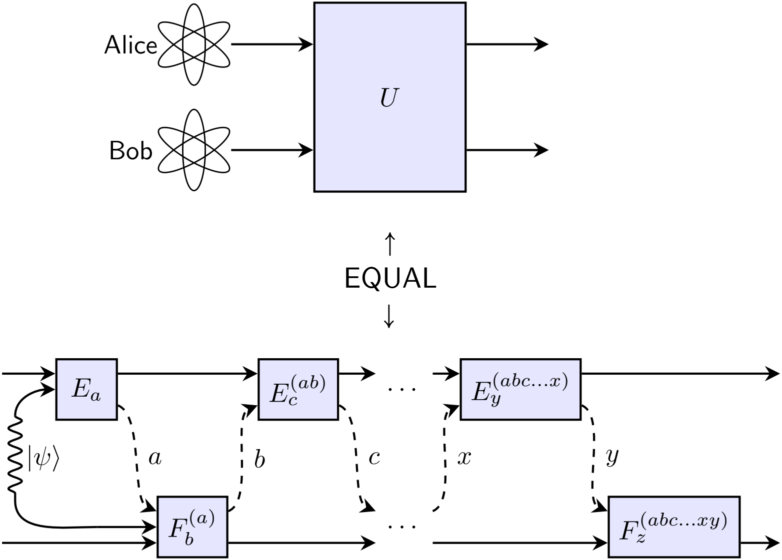
Bipartite unitaries



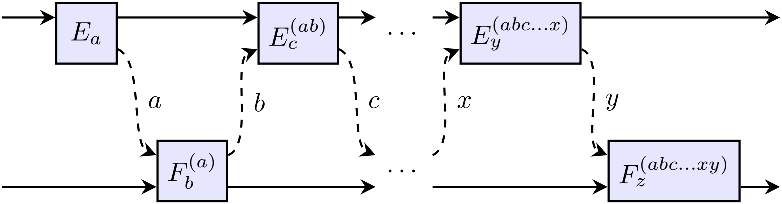
Bipartite unitaries



Bipartite unitaries

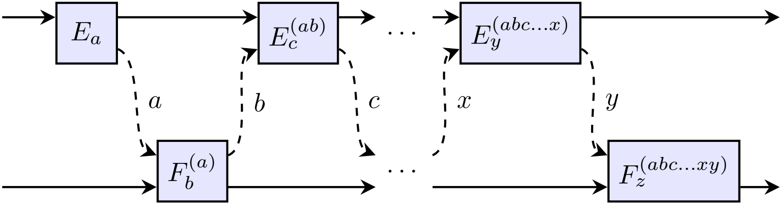


LOCC and SEP

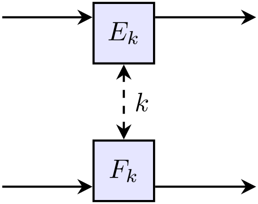


$$|\Phi\rangle \to \left(E_y^{(bc\ldots x)} \ldots E_c^{(ab)} E_a \otimes F_z^{(abc\ldots xy)} \ldots F_b^{(a)}\right)|\Phi\rangle$$

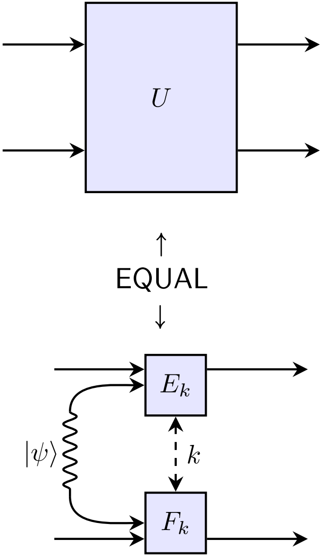
LOCC and SEP



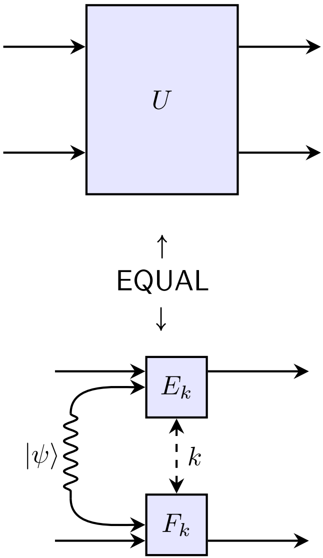
$$|\Phi\rangle \to \Big(\underbrace{E_y^{(bc\ldots x)} \ldots E_c^{(ab)} E_a}_{E_k} \otimes \underbrace{F_z^{(abc\ldots xy)} \ldots F_b^{(a)}}_{F_k} \Big) |\Phi\rangle$$



SEP implementing unitary



Results



- If Schmidt ranks of $|\psi\rangle$ and U are equal then $|\psi\rangle$ must be maximally entangled.
- Otherwise, $|\psi\rangle$ can sometimes have less entanglement (explicit numerical example of an LOCC protocol).

Chapter 3

Quantum interference as a resource for quantum speedup

Dan Stahlke, arXiv:1305.2186

Resource theory for quantum speedup

1) If a quantum circuit can be efficiently simulated (on a classical computer), then it doesn't exhibit quantum speedup.
2) If any quantum circuit that doesn't use resource X can be efficiently simulated, then resource X is necessary in order to have quantum speedup.

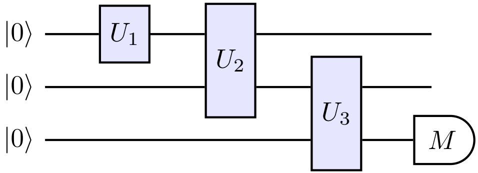
Resource theory for quantum speedup

1) If a quantum circuit can be efficiently simulated (on a classical computer), then it doesn't exhibit quantum speedup.
2) If any quantum circuit that doesn't use resource X can be efficiently simulated, then resource X is necessary in order to have quantum speedup.

Examples:

- $X =$ high Schmidt rank [Vid03, JL03]
- $X =$ non-Clifford operations [Got98]
- $X =$ negative values in Wigner representation [VFGE12, ME12]
- $X =$ large tree width [MS08, Joz06]
- $X =$ high interference producing capacity

Quantum circuits



Each U_i is unitary. Measurement M is a projector.
Expectation value of measurement:

$$\left\langle \psi \middle| U_1^\dagger U_2^\dagger U_3^\dagger M U_3 U_2 U_1 \middle| \psi \right\rangle$$

Shorthand: U_2 is $U_2 \otimes I$ for example.

Simulation

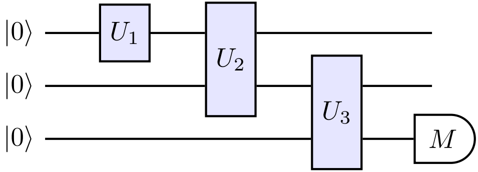
How to compute $\left\langle \psi \middle| U_1^\dagger U_2^\dagger U_3^\dagger M U_3 U_2 U_1 \middle| \psi \right\rangle$?

- Even writing down $|\psi\rangle$ takes $\Omega(2^n)$ memory.
- We could compute a sum over paths

$$\sum_{i,j,\dots,z} \psi_i^* (U_1^\dagger)_{ij} (U_2^\dagger)_{jk} \cdots (U_1)_{yz} \psi_z .$$

This takes $\Omega(2^n)$ time since each index runs over 2^n values.

Classical stochastic process



Each U_i is a stochastic matrix. Measurement M is an indicator function.

For example, $|0\rangle$ is heads, $|1\rangle$ is tails. $U_1 = \begin{pmatrix} 0.5 & 0 \\ 0.5 & 1 \end{pmatrix}$ means "if coin is heads, flip it again." U_2 could be "if first coin is heads, invert second coin."

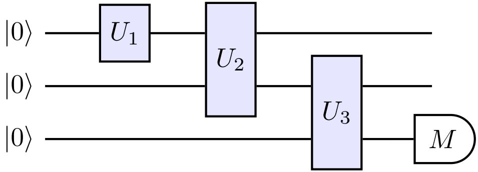
Expectation value of measurement is $M^T U_3 U_2 U_1 \psi$.

Simulation

How to compute $M^T U_3 U_2 U_1 \psi$?

- Again, writing down ψ takes $\Omega(2^n)$ memory.
- Again, a sum over paths takes $\Omega(2^n)$ time.

Simulation



But there is a simple and fast way to compute $M^T U_3 U_2 U_1 \psi$. Just flip the coins! After a few runs, the expectation value for the measurement can be estimated quite well.

Essentially, we are estimating $\sum_{ijkl} M_i (U_3)_{ij} (U_2)_{jk} (U_1)_{kl} \psi_l$ by randomly sampling paths.

So why can't the expectation value for the quantum circuit

$$\left\langle \psi \middle| U_1^\dagger U_2^\dagger U_3^\dagger M U_3 U_2 U_1 \middle| \psi \right\rangle$$

be estimated by sampling paths?

So why can't the expectation value for the quantum circuit

$$\left\langle \psi \middle| U_1^\dagger U_2^\dagger U_3^\dagger M U_3 U_2 U_1 \middle| \psi \right\rangle$$

be estimated by sampling paths?

Because of interference! The interference

$$\mathcal{I} = \sum_{i,j,\ldots,z} \left| \psi_i^* (U_1^\dagger)_{ij} (U_2^\dagger)_{jk} \cdots (U_1)_{yz} \psi_z \right|$$

may be very large.

If interference

$$\mathcal{I} = \sum_{i,j,\ldots,z} \left| \psi_i^* (U_1^\dagger)_{ij} (U_2^\dagger)_{jk} \cdots (U_1)_{yz} \psi_z \right|$$

is small, then the expectation value can be efficiently computed as long as you know how to find the paths with the greatest weight.

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But I don't know how to find such paths (this is an open question). On the other hand, I can simulate the circuit if each operator is not capable of producing much interference.

If interference

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But I don't know how to find such paths (this is an open question). On the other hand, I can simulate the circuit if each operator is not capable of producing much interference.

Interference producing capacity:

$$\mathcal{I}_{\mathsf{max}}(U_1) = \|\mathrm{abs}(U_1)\|_2.$$

Main result

I can estimate

$$\left\langle\psi\middle|U_1^\dagger U_2^\dagger U_3^\dagger M U_3 U_2 U_1\middle|\psi\right\rangle$$

to accuracy ϵ in time

$$O\left(\epsilon^{-2}\mathcal{I}_{\mathsf{max}}(M)^2\prod_i \mathcal{I}_{\mathsf{max}}(U_i)^4\right).$$

Operator	$\mathcal{I}_{\mathsf{max}}$
Fourier or Hadamard transform	$2^{n/2}$
Haar wavelet transform	$\sqrt{1+n}$
Grover reflection	$\mathcal{I}_{\mathsf{max}} \to 3$ as $n \to \infty$
Permutation	1
Pauli matrices	1

Other results

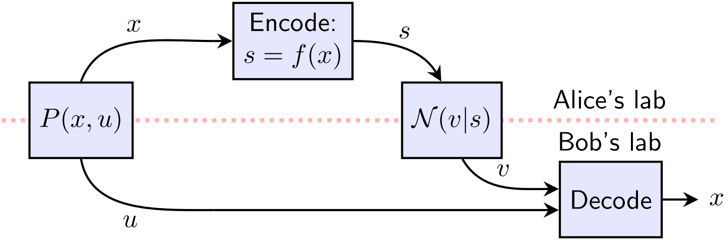
- Generalizes to other ℓ^p norms. Taking $p = 1$ generalizes recent work regarding "mana" [VMGE14].
- No quantum advantage for communication problems if $\prod \mathcal{I}_{\mathsf{max}}$ is low, unless the number of rounds is bounded.
- $\mathcal{I}_{\mathsf{max}}$ is the first continuous-valued quantity linked to quantum speedup.

Chapter 4

Bounds on Entanglement Assisted Source-channel Coding via the Lovász ϑ Number and its Variants

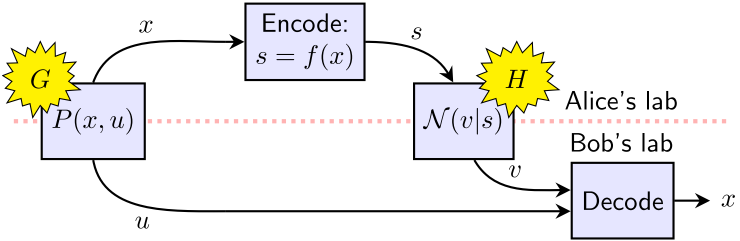
Toby Cubitt, Laura Mančinska, David Roberson, Simone Severini, Dan Stahlke, and Andreas Winter,
arXiv:1310.7120

- Alice wants to send message x to Bob, using a noisy channel.
- Bob already has some side information u regarding Alice's message x.
- They both know which (x, u) pairs are possible, and know the details of the channel noise.
- Protocol must succeed with 100% certainty!



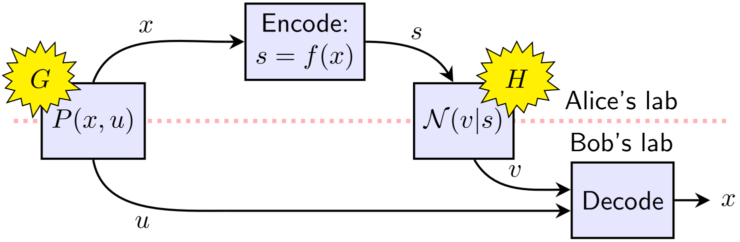
From [NTR06]:

- Graph G has edges between x and x' if side information doesn't distinguish between these.
- Graph H has edges between s and s' if the channel never maps these inputs to the same output.
- G represents info that needs to be sent, H represents info that makes it through the channel.

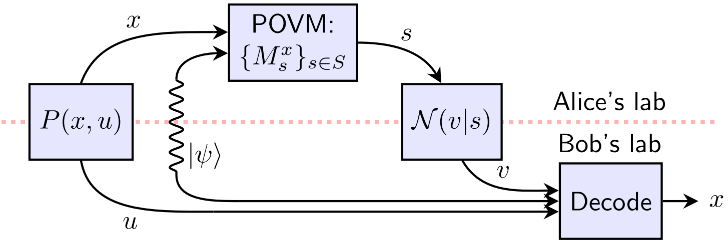


From [NTR06]:

- Encoding must map info that needs to be sent into codewords that make it through the channel.
- $x \sim_G y \implies f(x) \sim_H f(y)$.
- This is a *homomorphism* $G \to H$.



Entanglement assistance



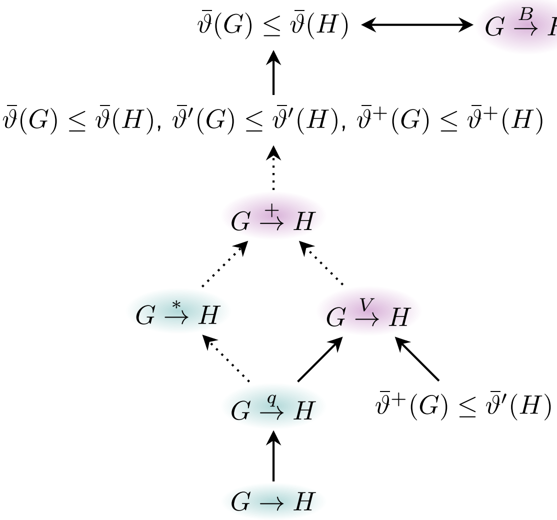
- Without entanglement assistance, source-channel coding is possible when $G \to H$.
- With entanglement assistance, write $G \xrightarrow{*} H$ when a protocol is possible [BBL+13].
- A closely related concept, $G \xrightarrow{q} H$ was studied in [RM12].

Our contribution

- $G \overset{*}{\to} H$ and $G \overset{q}{\to} H$ are very difficult to compute.
- We study three related semidefinite relaxations, which we call $G \overset{B}{\to} H$, $G \overset{+}{\to} H$, and $G \overset{V}{\to} H$.
- We found these to be closely related to the Lovász ϑ function.

(it seems all paths lead to ϑ!) - Goemans [Goe97]

Results


$\bar{\vartheta}(G) \leq \bar{\vartheta}(H)$
$G \xrightarrow{B} H$
$\bar{\vartheta}(G) \leq \bar{\vartheta}(H)$, $\bar{\vartheta}'(G) \leq \bar{\vartheta}'(H)$, $\bar{\vartheta}^+(G) \leq \bar{\vartheta}^+(H)$
$G \xrightarrow{+} H$
$G \xrightarrow{*} H$
$G \xrightarrow{V} H$
$G \xrightarrow{q} H$
$\bar{\vartheta}^+(G) \leq \bar{\vartheta}'(H)$
$G \rightarrow H$

Results

Open questions answered:

- A parameter defined by Beigi [Bei10] is exactly equal to $\lceil \bar{\vartheta} \rceil$.
- There is a gap between quantum chromatic number and its semidefinite relaxation given in [PT13].

New bounds:

- Cost rate is at least $\log \bar{\vartheta}(G) / \log \bar{\vartheta}(H)$.
- One-shot entanglement assisted capacity is at most $\bar{\vartheta}'(H)$.

Reproduced/generalized previously known results:

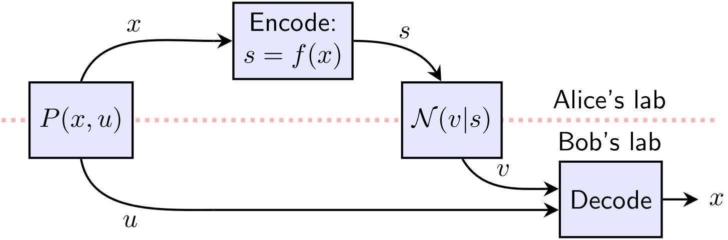
- Entanglement assisted chromatic number is at least $\bar{\vartheta}^{+}(G)$ [BBL+13].
- Generalized theorem 2.7 of [GL08] regarding $\bar{\vartheta}$, $\bar{\vartheta}'$, $\bar{\vartheta}^{+}$.

Chapter 5

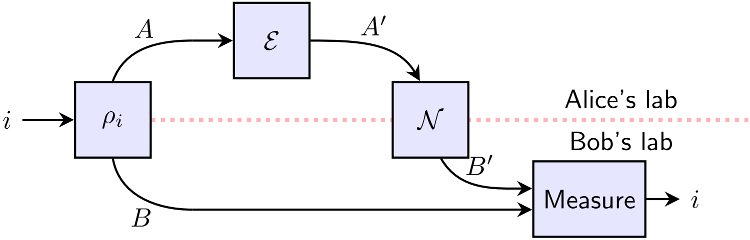
Quantum source-channel coding and non-commutative graph theory

Dan Stahlke, arXiv:1405.5254

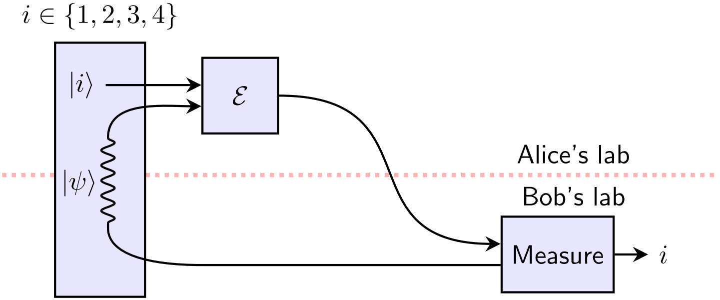
Classical source-channel coding



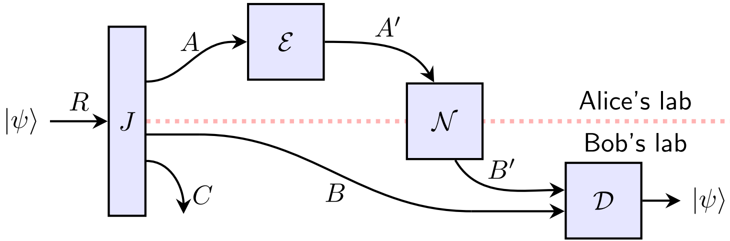
(Discrete) quantum source-channel coding



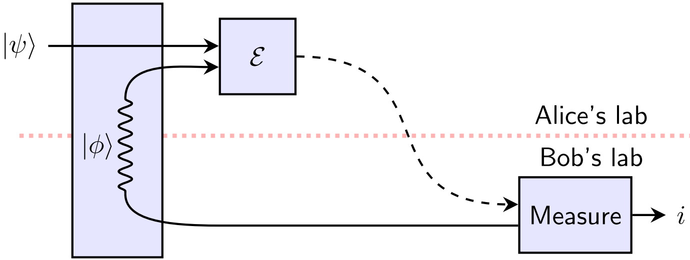
Example: Dense coding



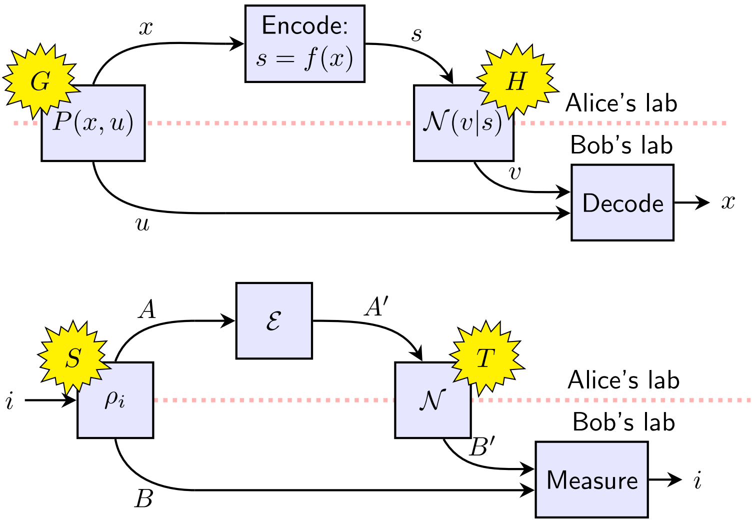
Coherent quantum source-channel coding



Example: Teleportation



Idea



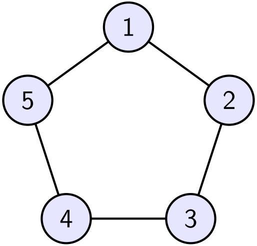
G
x
Encode:
$s = f(x)$
s
H
$P(x, u)$
$\mathcal{N}(v|s)$
Alice's lab
Bob's lab
v
Decode
x
u
S
A
$\mathcal{E}$
A'
T
i
ρ_i
$\mathcal{N}$
Alice's lab
Bob's lab
B'
Measure
i
B

Non-commutative graphs

For a (classical) graph G, one can consider the operator subspace

$$S = \mathrm{span}\{|i\rangle\langle j| : i \sim j\}.$$

For example:

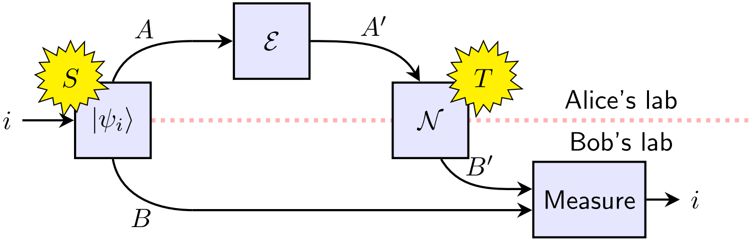


$$S = \mathrm{span}\begin{pmatrix} 0 & * & 0 & 0 & * \\ * & 0 & * & 0 & 0 \\ 0 & * & 0 & * & 0 \\ 0 & 0 & * & 0 & * \\ * & 0 & 0 & * & 0 \end{pmatrix}$$

Non-commutative graphs

In general, one can consider operator subspaces that are *not* of the form $\mathrm{span}\{|i\rangle\langle j| : i \sim j\}$ (but still demanding $M^\dagger \in S$ when $M \in S$). These are called *non-commutative graphs*.

The Lovász ϑ number has been defined for these, and it upper bounds the entanglement assisted zero-error capacity of a quantum channel [DSW13].



$$S = \operatorname{span}\{\operatorname{Tr}_B(|\psi_i\rangle\langle\psi_j|) : i \neq j\}$$

$$T = \operatorname{span}\{N_i^\dagger N_j\}^\perp$$ [DSW13]

I extend the notion of graph homomorphism to non-commutative graphs. A source-channel coding protocol is possible if and only if $S \to T$.

Results

- I define a homomorphism for non-commutative graphs. From this follows, for instance, defining a chromatic number for such graphs.
- I generalize the Schrijver $\bar{\vartheta}'$ and Szegedy $\bar{\vartheta}^+$ numbers for non-commutative graphs.
- These, and the Lovász $\bar{\vartheta}$, are homomorphism monotones. E.g. $S \to T \implies \bar{\vartheta}(S) \leq \bar{\vartheta}(T)$.
- These quantities provide new bounds on one-shot zero-error channel capacity and on measurement of bipartite states using one-way communication.

Results

- With entanglement assistance, the condition is $S \otimes \Lambda \to T$.
- The Lovász number is monotone also with entanglement assistance, but the Schrijver and Szegedy generalizations are not (contrary to the classical case).
- Curiously, one of my Schrijver variants counts maximally entangled states as being less valuable than non-maximally entangled states.
- I use this to construct a channel which can transmit quantum information when assisted by a non-maximally entangled state, but cannot transmit any error-free information at all when assisted by a maximally entangled state.

Open question

Graph theory has been a deep and lucrative topic.

Are there gems to be found in the theory of non-commutative graphs?

Thanks for listening.

Questions?

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