Quantum interference as a resource for quantum speedup [arXiv:1305.2186]

What is the source of the quantum speedup?

Quantum computers can sometimes vastly outperform classical computers (e.g. Shor or Grover). What is the "secret sauce" that gives quantum computers their advantage?

✗ **Is it entanglement/discord?**
Universal quantum computation can be achieved with an arbitrarily low amount of entanglement or discord [2] (although Schmidt rank must be high [3]).

✗ **Is it the exponentially large dimension of the Hilbert space?**
This is not unique to qubits: probability distributions over n bits also have dimension 2^n.

> "To describe a state of n particles, we need to write down an exponentially long vector of exponentially small numbers, which themselves vary continuously. Moreover, the instant we measure a particle, we "collapse" the vector that describes its state – and not only that, but possibly the state of another particle on the opposite side of the universe. Quick, what theory have I just described? The answer is classical probability theory." - Aaronson (quant-ph/0507242)

✓ **Is it quantum interference?**
This is something that probability distributions don't have. It is often mentioned, qualitatively, as being the source of quantum speedup. I quantify this.

Main result

Consider a quantum circuit that ends with a two-outcome (yes/no) measurement. Limiting to a single unitary for simplicity of presentation, the goal is to estimate $\langle\psi|U^\dagger MU|\psi\rangle$. This can be written as a special case of a more general form,

$$\langle\psi|U^\dagger MU|\psi\rangle = \langle\psi|ABC|\phi\rangle,$$

which can be computed via a sum over Feynman-like paths in the computational basis:

$$\langle\psi|ABC|\phi\rangle = \sum \psi_i^* A_{ij} B_{jk} C_{kl} \phi_l.$$

We define a ***measure of interference*** via a sum over absolute values:

$$\mathcal{I} = \sum \left|\psi_i^* A_{ij} B_{jk} C_{kl} \phi_l\right|.$$

The ***interference producing capacity*** of an operator is the maximum amount of interference it can produce, which ends up being

$$\mathcal{I}_{\max}(A) = \|\mathrm{abs}(A)\|,$$

where $\mathrm{abs}(A)_{ij} := |A_{ij}|$ and $\|\cdot\|$ is the maximum singular value. Modulo some technical details, we have the following theorem:

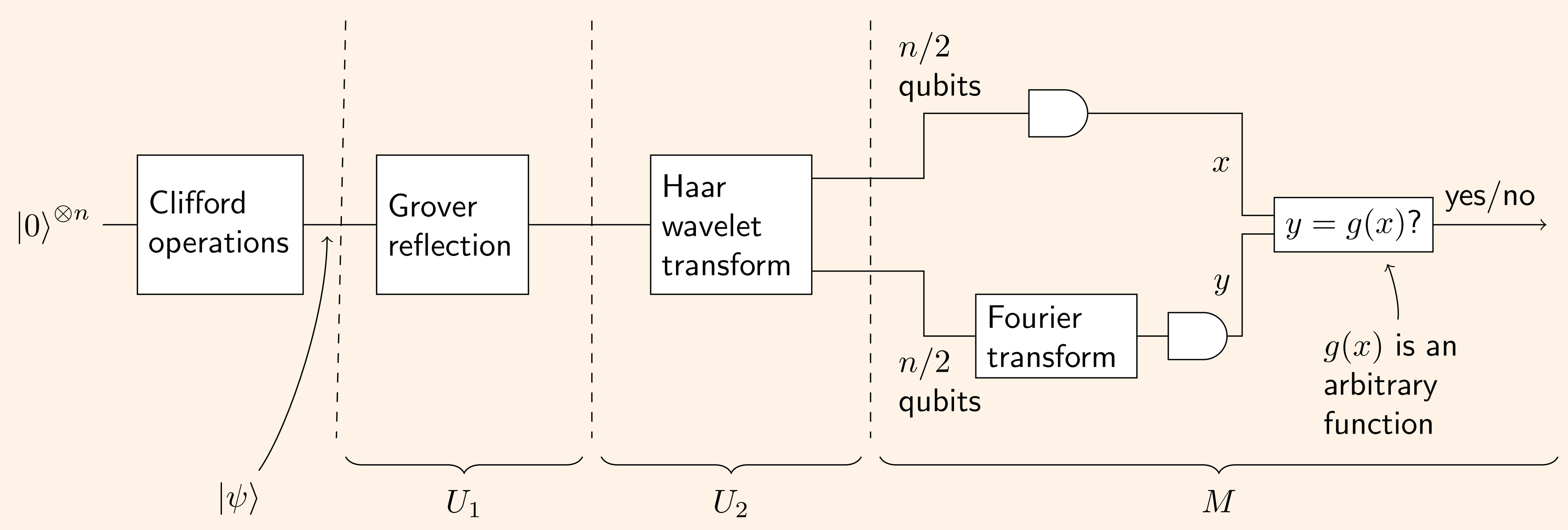
Theorem: The value of $\langle\psi|A\ldots Z|\phi\rangle$ can be estimated to accuracy $\pm\epsilon$ on a classical computer in time $O\left(\epsilon^{-2}\mathcal{I}_{\max}(A)^2 \ldots \mathcal{I}_{\max}(Z)^2\right)$.

The simulation involves sampling paths chosen according to a convex combination of two Markov chains, one going left-to-right ($\psi \to A \to B \to \phi$) and the other going right-to-left. This was inspired by, and can be seen as an extension of, [4] which dealt with sparse matrices.

Open Question: Can we do it in time $O\left(\epsilon^{-2}\mathcal{I}^2\right)$?

Example #1

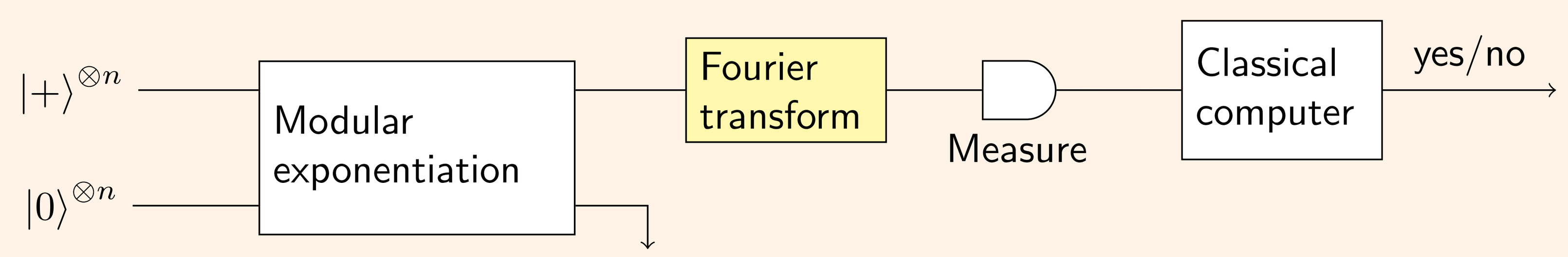
The following circuit appears, at first sight, to use much quantum magic. However, it can be efficiently simulated.



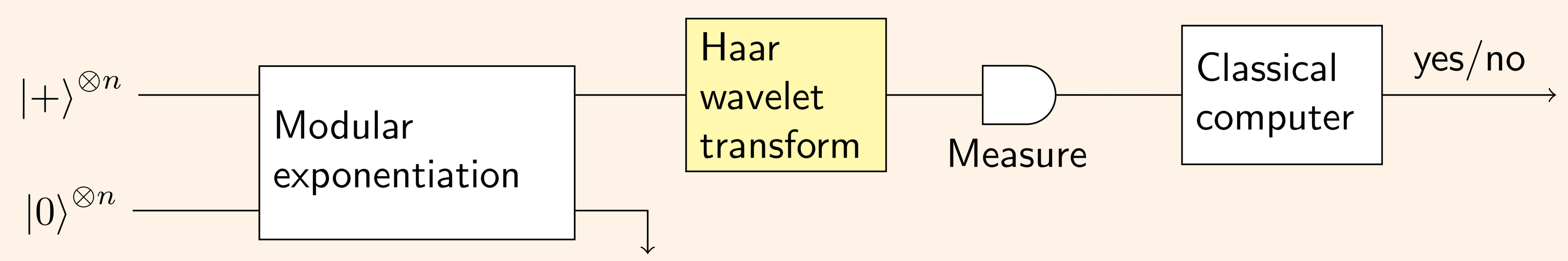
The output of the Clifford circuit is considered to be the initial state $|\psi\rangle$. The Grover reflection (U_1) and Haar wavelet transform (U_2) have low interference producing capacity, as does the measurement at the end (M is taken to be the entire last block).
Note: The Clifford operators are Paulis, Hadamard, $\pi/2$ phase gate, and CNOT.

Example #2

I can't efficiently simulate Shor's algorithm because the Fourier transform has high interference producing capacity:



However, replacing the Fourier with a Haar wavelet transform yields an efficiently simulatable circuit (that no longer does anything useful). The Haar wavelet transform has low interference producing capacity because the high frequency wavelets are spatially localized:



Note: the modular exponentiation and the measurement/post-processing are classical and so have $\mathcal{I}_{\max} = 1$.

What can be efficiently simulated

Quantum circuits can be efficiently simulated using this technique if:

- You can efficiently simulate measurements of the initial state $|\psi\rangle$ in the computational basis, and also compute the value of individual entries, $\langle i|\psi\rangle$. This includes the CT (computationally tractable) states of [4].
- The unitaries and final measurement of the quantum circuit have low interference producing capacity and meet a certain requirement for efficient computability.
- Note: as it involves a product of interference producing capacities, the simulation cost rises exponentially with the length of the circuit (number of successive unitaries).

I can simulate the following types of unitaries and measurements:

- ECS (efficiently computable sparse) matrices from [4], such as permutations (i.e. classical computations) and Pauli operators
- Grover reflections: $I - 2|+\rangle^{\otimes n}\langle +|^{\otimes n}$
- Haar wavelet transform
- Short time/low energy Hamiltonian evolutions (at cost $e^{\|\mathrm{abs}(H)\|t}$)
- Rank one projectors and diagonal projectors
- Sums and products of efficiently simulatable operators
- Block diagonal operators where each block is efficiently simulatable

References

[1] D. Stahlke, "Quantum interference as a resource for quantum speedup," 2013, arXiv:1305.2186.

[2] M. V. d. Nest, "Universal quantum computation with little entanglement," 2012, arXiv:1204.3107.

[3] G. Vidal, "Efficient classical simulation of slightly entangled quantum computations," *Phys. Rev. Lett.*, vol. 91, p. 147902, Oct 2003.

[4] M. Van den Nest, "Simulating quantum computers with probabilistic methods," *Quantum Information and Computation*, vol. 11, no. 9&10, pp. 0784–0812, 2011.

Footnote

This is what the Haar wavelet transform looks like on three qubits:

$$\begin{bmatrix}
\frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} \\
\frac{1}{\sqrt8} & \frac{-1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{-1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{-1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{-1}{\sqrt8} \\
\frac{1}{\sqrt4} & 0 & \frac{-1}{\sqrt4} & 0 & \frac{1}{\sqrt4} & 0 & \frac{-1}{\sqrt4} & 0 \\
0 & \frac{1}{\sqrt4} & 0 & \frac{-1}{\sqrt4} & 0 & \frac{1}{\sqrt4} & 0 & \frac{-1}{\sqrt4} \\
\frac{1}{\sqrt2} & 0 & 0 & 0 & \frac{-1}{\sqrt2} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt2} & 0 & 0 & 0 & \frac{-1}{\sqrt2} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt2} & 0 & 0 & 0 & \frac{-1}{\sqrt2} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt2} & 0 & 0 & 0 & \frac{-1}{\sqrt2}
\end{bmatrix}$$

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