# Bounds on Entanglement Assisted Source-channel Coding via the Lovász $\vartheta$ Number and its Variants [arXiv:1310.7120]

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## Background: Source-channel coding

Suppose that Alice and Bob receive a pair $(x, u)$ with probability $Q(x, u)$. Alice wishes to send $x$ to Bob, using a noisy classical channel $\mathcal{N}$, such that Bob can determine $x$ with zero chance of error. Without making use of entanglement, this is known [2] to be possible iff there is a graph homomorphism $G \to H$ between the graphs

$$x \sim_G y \iff \exists u \in U \text{ such that } Q(x, u)Q(y, u) \neq 0.$$
$$s \sim_H t \iff \mathcal{N}(v|s)\mathcal{N}(v|t) = 0 \text{ for all } v \in V.$$

Basically, $G$ represents the information that needs to be sent and $H$ represents the information that survives the channel. A homomorphism $G \to H$ ensures that the needed information makes it through the channel intact.

If Alice and Bob share an entangled state they can use the strategy depicted in the figure, which is described in greater detail in [3].
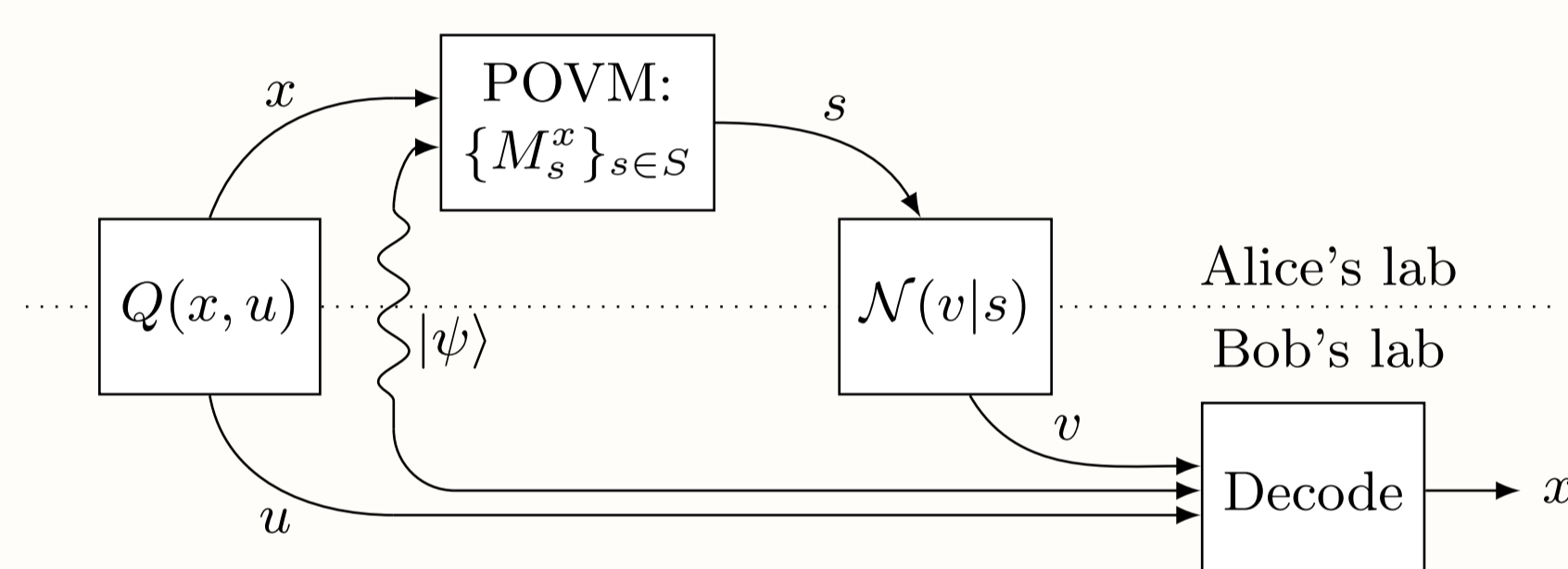


After Alice's measurement, Bob's half of the entanglement resource is in the state

$$\rho_s^x = \mathsf{Tr}_A\{(M_s^x \otimes I)|\psi\rangle\langle\psi|\}.$$

An error free decoding operation exists for Bob if and only if these states are orthogonal for every $x \in X$ consistent with the information in Bob's possession (i.e. $u$ and $v$) [3].

$$\rho_s^x \perp \rho_t^y \text{ for all } x \sim_G y \text{ and } s \not\sim_H t.$$

If such $|\psi\rangle$ and $\{M_s^x\}$ exist, we say there is an *entanglement assisted homomorphism* $G \xrightarrow{*} H$.

The largest $n$ such that $K_n \xrightarrow{*} H$ is the *entanglement assisted independence number* $\alpha^*(\overline{H})$, the largest number of error-free codewords that can be sent through channel $\mathcal{N}$. The smallest $n$ such that $G \xrightarrow{*} K_n$ is the *entanglement assisted chromatic number* $\chi^*(G)$, the size of the smallest channel that Alice could use to convey $x$ from source $Q$.

## Background: Quantum Homomorphism

A related concept, *quantum homomorphism*, is defined in terms of a nonlocality game. Suppose that Alice and Bob share an entangled state but are not allowed to communicate. Alice is told a vertex $x \in V(G)$ and Bob is told $y \in V(G)$. Alice and Bob then reply with $s, t \in V(H)$. If $x = y$ then their answers must be the same. If $x \sim_G y$ then their answers must satisfy $s \sim_H t$. If such a game can be won with certainty, then there is a quantum homomorphism $G \xrightarrow{q} H$. It can be shown that projective measurements on a maximally entangled state suffice here.
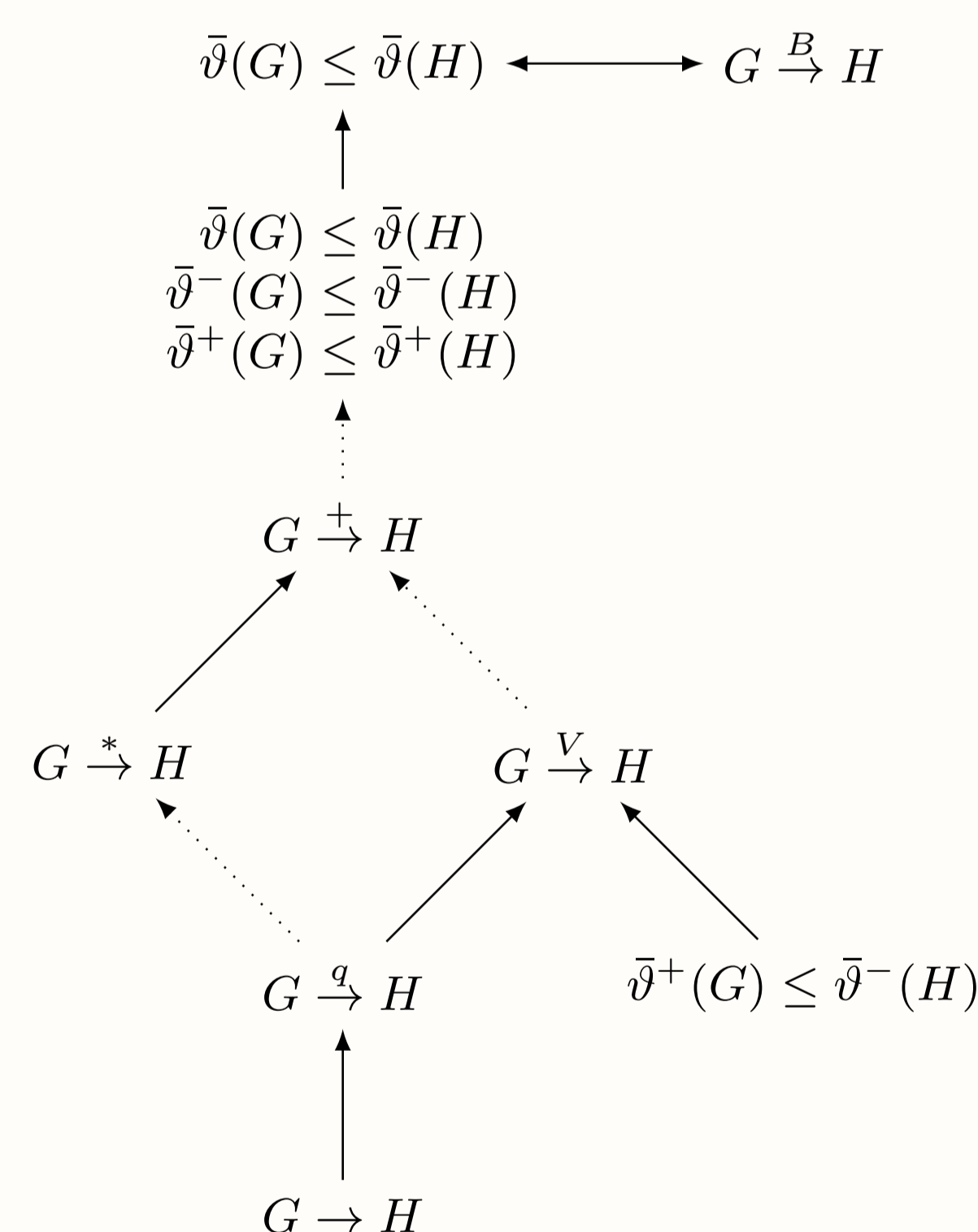
## Semidefinite relaxation

We consider several relaxations of $G \xrightarrow{*} H$, and relate these to monotonicity of the Lovász $\vartheta$ function and its variants.

- Say $G \xrightarrow{+} H$ if there is a probability distribution $P(s, t|x, y)$ such that
  (i) $P_{sx;ty}$ is a positive semidefinite matrix
  (ii) $P(s \sim_H t | x \sim_G y) = 1$
- Say $G \xrightarrow{V} H$ if there is a probability distribution $P(s, t|x, y)$ such that
  (i) $P_{sx;ty}$ is a positive semidefinite matrix
  (ii) $P(s \sim_H t | x \sim_G y) = 1$
  (iii) $P(s = t | x = y) = 1$
- Say $G \xrightarrow{B} H$ if there is a quasi-probability distribution $P(s, t|x, y)$ (i.e. allowing negative probabilities) such that
  (i) $P_{sx;ty}$ is a positive semidefinite matrix
  (ii) $P(s \sim_H t | x \sim_G y) = 1$
  (iii) $P(s = t | x = y) = 1$

Note: if $P_{sx;ty}$ is required to be *completely PSD* [4] rather than PSD in the above definitions, then $G \xrightarrow{+} H$ becomes $G \xrightarrow{*} H$ and $G \xrightarrow{V} H$ becomes $G \xrightarrow{q} H$.

## Theorem



In the figure, arrows mean one condition implies the other. Dotted arrows mean we don't know whether the reverse implication holds. $\bar{\vartheta}(G)$, $\bar{\vartheta}^-(G)$, and $\bar{\vartheta}^+(G)$ are the Lovász, Schrijver, and Szegedy numbers of the complement graph $\overline{G}$.

$$\bar{\vartheta}(G) = \max\{\langle B, J\rangle : B \succeq 0, \mathsf{Tr}B = 1,$$
$$B_{ij} = 0 \text{ for } i \not\sim j, i \neq j\}, \quad (1)$$
$$\bar{\vartheta}(G) = \min\{\lambda : \exists Z \succeq 0, Z_{ii} = \lambda - 1,$$
$$Z_{ij} = -1 \text{ for } i \sim j\}, \quad (2)$$
$$\bar{\vartheta}^-(G) = \min\{\lambda : \exists Z \succeq 0, Z_{ii} = \lambda - 1,$$
$$Z_{ij} \leq -1 \text{ for } i \sim j\}.$$
$$\bar{\vartheta}^+(G) = \min\{\lambda : \exists Z \succeq 0, Z_{ii} = \lambda - 1,$$
$$Z_{ij} = -1 \text{ for } i \sim j,$$
$$Z_{ij} \geq -1 \text{ for all } i, j\}.$$

### Proof

- $G \xrightarrow{*} H \implies G \xrightarrow{+} H$: Let $P(s, t|x, y) = \langle \rho_s^x, \rho_t^y\rangle / \langle \rho, \rho\rangle$.
- $G \xrightarrow{*} H \implies G \xrightarrow{B} H$: W.l.o.g. assume the $M_s^x$ are projectors. Let $P(s, t|x, y) = \langle \psi | M_s^x M_t^y \otimes I |\psi\rangle$.
- $G \xrightarrow{B} H \implies \bar{\vartheta}(G) \leq \bar{\vartheta}(H)$: Let $Z_{st}$ be feasible for (2) for $H$. Then $Y_{xy} = \sum_{st} Z_{st} P(s, t|x, y)$ is feasible for $G$ with the same value.
- $\bar{\vartheta}(G) \leq \bar{\vartheta}(H) \implies G \xrightarrow{B} H$: Let $B$ be feasible for (1) for $H$ and $Z$ be feasible for (2) for $G$, both with value $\lambda = \bar{\vartheta}(H)$. Let matrix $D$ consist of the diagonal entries of $B$. Define $P(s, t|x, y) = \lambda^{-1}\left[B_{st} + (\lambda - 1)^{-1}Z_{xy}(\lambda D_{st} - B_{st})\right]$.

## Corollary

The one-shot zero-error entanglement assisted capacity of a channel, $\alpha^*(\overline{H})$, is upper bounded by $\bar{\vartheta}^-(H)$.

## Corollary

The entanglement assisted chromatic number, $\chi^*(G)$, is lower bounded by $\bar{\vartheta}^+(G)$. This was already shown in [3].

## Corollary

The entanglement assisted cost rate (asymptotic number of channel uses needed per source instance) is bounded as follows:

$$\eta^*(G, \overline{H}) \geq \frac{\log \bar{\vartheta}(G)}{\log \bar{\vartheta}(H)}.$$

## Corollary

Beigi [5] introduced a quantity $\beta(\overline{H})$ defined as the largest $n$ such that there exist vectors $|w\rangle \neq 0$ and $|w_s^x\rangle$ with $x \in \{1, \ldots, n\}$ and $s \in V(H)$ which satisfy

$$\sum_s |w_s^x\rangle = |w\rangle$$
$$\langle w_s^x | w_t^y\rangle = 0 \text{ for all } x \neq y, s \not\sim_H t$$
$$\langle w_s^x | w_t^x\rangle = 0 \text{ for all } s \neq t.$$

Beigi showed that $\alpha^*(\overline{H}) \leq \beta(\overline{H}) \leq \lfloor\bar{\vartheta}(H)\rfloor$ and asked whether $\beta(\overline{H}) = \lfloor\bar{\vartheta}(H)\rfloor$. We answer this in the affirmative since $\beta(\overline{H})$ is just the largest $n$ such that $K_n \xrightarrow{B} H$ (let $P_{sx;ty}$ be the Gram matrix of $|w_s^x\rangle$).

## Corollary

In [6] a quantity $\chi_{\text{vect}}(G)$ was introduced which, in our terminology, is the largest $n$ such that $K_n \xrightarrow{V} G$. The question was posed whether there is a graph such that $\chi_{\text{vect}}(G) < \chi_q(G)$. We show that $\chi_{\text{vect}}(G) = \lceil\bar{\vartheta}^+(G)\rceil$ and that the graph $G = C_5 * K_3$ satisfies $\chi_{\text{vect}}(G) < \chi_q(G)$.

## References

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