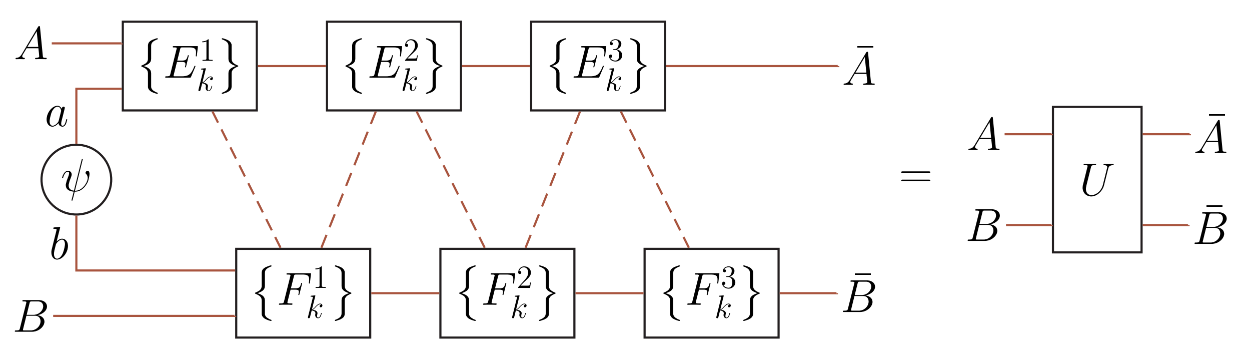
Entanglement requirements for implementing bipartite unitary operations[1]

Introduction

- Suppose that Alice and Bob want to cooperatively carry out a quantum computing operation (this is called a *bipartite operation*). Alice and Bob are in different laboratories and are unable to send quantum material back and forth, but they can communicate over the telephone (or the Internet). This is known as *local operations and classical communication* (LOCC). Are they able to carry out their desired operation?
- Given no other resources the answer is no, they cannot. However, if they happen to be in possession of a pair of entangled particles (e.g. an EPR pair) then such a cooperative computation is possible. The object of my research has been to investigate how much entanglement is needed.
- Why care about such a thing? The Alice and Bob scenario could for instance arise in the case of a quantum computer. It may be that in the future we will have a quantum microprocessor which is divided into several subunits, with classical (non-quantum) communication between the subunits being cheap and transmission of entangled states between the subunits being expensive. In this case "Alice" and "Bob" represent the subunits of the processor.
- Another situation involving Alice and Bob is with quantum cryptography, in which the two parties wish to use quantum mechanics to communicate or perform some other task in a way that is secure from eavesdropping or tampering. This is not directly related to the work I have been doing, but it is indirectly related in that it is concerned with the investigation of two party (bipartite) systems.
- Alice and Bob's cooperative task shown as a diagram. A and B are Alice and Bob's input states, $\bar{A}$ and $\bar{B}$ are their output states. ψ is the preshared entanglement resource that allows them to perform their desired operation (U) using only classical communication (the dotted lines). The E and F blocks represent local (Alice for E, Bob for F) quantum operations. Time progresses from left to right.



Results

- The result of this line of research is that I have found a lower bound on the amount of preshared entanglement needed in order to carry out an arbitrary unitary (i.e. reversible) operation. Also I have found that, somewhat counterintuitively, using an entanglement resource of a larger than necessary dimension allows one to use less total entanglement.
- This lower bound on entanglement was known for dimension two [2]. I have found a much simpler proof that is applicable to arbitrary dimensions, using a somewhat novel technique based upon atemporal diagrams [3].
- Our paper is in arXiv [1].

Separable operations

- Any LOCC protocol (such as was described in the introduction) can be transformed into a *separable operation* (SEP). This is nice because SEP is much easier to deal with mathematically.
- A separable operation consists of sets of operators $\{E_k\}$ and $\{F_k\}$ satisfying the closure condition

$$\sum_k (E_k \otimes F_k)^\dagger (E_k \otimes F_k) = I_A \otimes I_a \otimes I_B \otimes I_b. \quad (1)$$

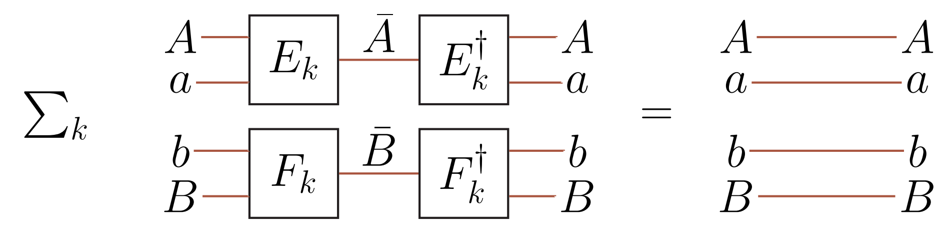
- In the case of interest here, Alice and Bob wish to deterministically (i.e. without chance of failure) carry out a unitary operation, consuming a shared entanglement resource $|\psi\rangle$ in the process. Such a separable operation must satisfy

$$(E_k \otimes F_k)\,|\psi\rangle = \alpha_k U \quad (2)$$

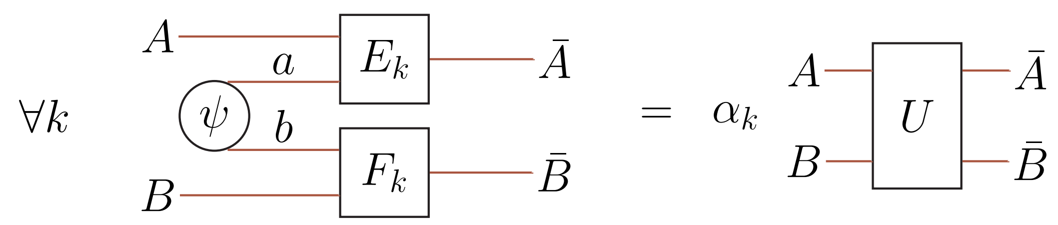
for all k, where α_k is a complex number.

Atemporal diagrams

- Equations such as (1) and (2) that involve a large number of systems can be hard to grasp. Atemporal diagrams [3] offer a simple and convenient way to visualize such equations. Any linear operator may be considered as just a matrix of numbers (or a larger block of numbers for tensors of higher rank). Each operator is drawn as a box or circle, and tensor contractions (operator multiplication) are represented as lines between the boxes.
- This is similar to the (temporal) diagrams shown to the left, with the difference being that we no longer imagine the horizontal position as representing time; all that matters is the nodes and the connections between them.
- In this scheme, (1) is depicted as

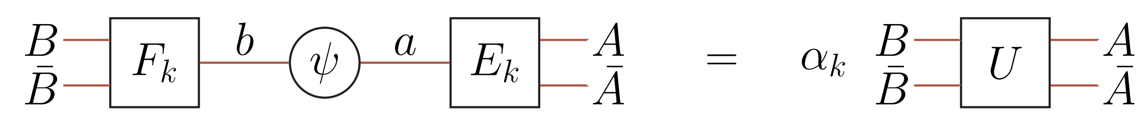


and (2) is depicted as

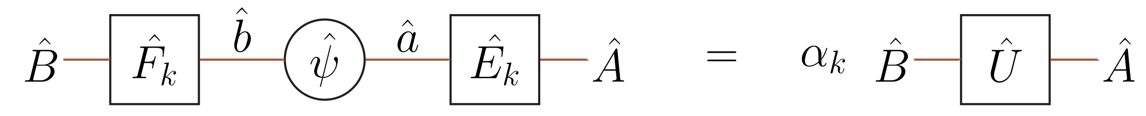


Atemporal diagrams

- The power of atemporal diagrams is that it is possible to understand complex operations from an intuitive point of view. Lets draw that last diagram in a more suggestive form.



- This is starting to look like a simple matrix equation (i.e. each operator takes one space in and puts one space out). Lets combine the $A \otimes \bar{A}$ space into a single space named $\hat{A}$ and do the same with the B spaces.



- Note that the resource state $|\psi\rangle$ is now being viewed as an operator. This is known as map-state duality. Note also that the operator $\hat{U}$ as shown in the last picture is a partial transpose of the original unitary U. The matrix rank of these two operators is the *Schmidt rank* of $|\psi\rangle$ and U. From basic linear algebra, it is apparent from the last picture that the Schmidt rank of $|\psi\rangle$ must be at least as great as the Schmidt rank of U. This proves part (a) of the theorem stated below.

Theorem

Suppose that a unitary operator U is implemented by a separable operation that makes use of the entanglement resource $|\psi\rangle$. Then
(a) The Schmidt rank D_ψ of $|\psi\rangle$ is greater than or equal to the Schmidt rank D_U of U.
(b) If the Schmidt ranks are equal, then $|\psi\rangle$ must be a uniformly (maximally) entangled state: all the nonzero Schmidt coefficients are the same.
(c) There are cases with $D_U = 2$, $D_\psi = 3$ where $|\psi\rangle$ may have less than one ebit of entanglement.

Example

- Consider a controlled phase operation: Alice and Bob's input states consist of qubits (spin-$\frac{1}{2}$ particles). If Alice and Bob's inputs are both spin down then the phase of the state is altered by an amount $e^{i\phi}$, otherwise nothing happens.
- The unitary in this case is

$$U = |0\rangle_A \langle 0|_A \otimes |0\rangle_B \langle 0|_B + |0\rangle_A \langle 0|_A \otimes |1\rangle_B \langle 1|_B + |1\rangle_A \langle 1|_A \otimes |0\rangle_B \langle 0|_B + e^{i\phi} |1\rangle_A \langle 1|_A \otimes |1\rangle_B \langle 1|_B \quad (3)$$

- Or, as a matrix,

$$U = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{array}\right) \quad (4)$$

- The Schmidt rank of U is 2, so the Schmidt rank of the entanglement resource must be at least 2. If the resource is exactly rank 2 (a qubit pair) then it must be fully entangled (an EPR pair). However, if the phase ϕ is small enough then by using a rank 3 resource (qutrits) it is possible to get by with a bit less entanglement than an EPR pair would have. I have found numerically a protocol with $\phi = 0.08\pi$ that uses 0.92 bits of entanglement (an EPR pair is 1 bit).

Proof of (b)

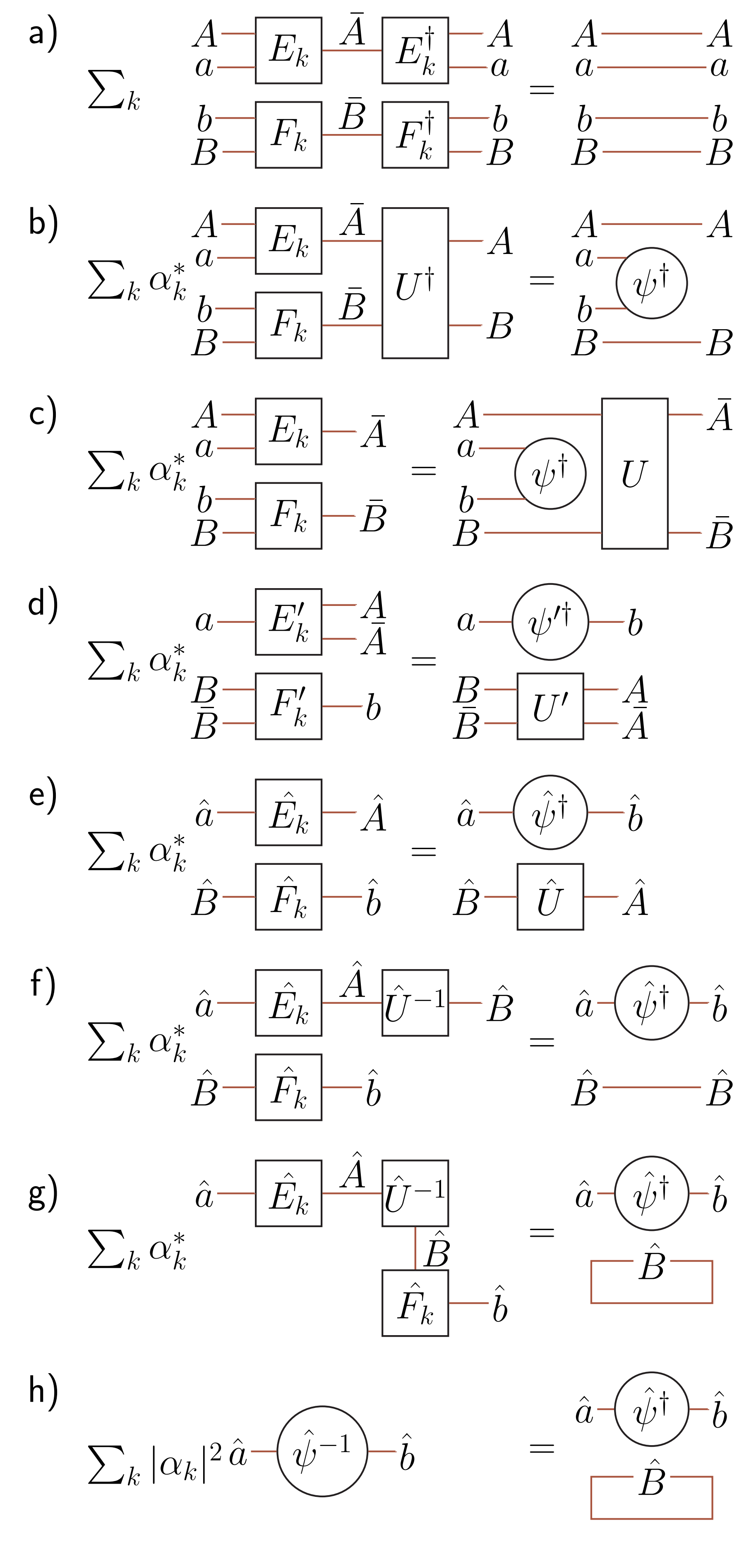


Figure: (a) The closure condition (1). (b) Act with $\langle\psi|$ and apply (2). (c) Multiply on the right by U. (d) Rearrange. (e) Combine subspaces. (f) Multiply by $\hat{U}^{-1}$. (g) Trace over $\mathcal{H}_{\hat{B}}$. (h) Substitute $U^{-1} = (\hat{E}_k \hat{\psi} \hat{F}_k)^{-1}$.

$$\text{Then, } \sum_k |\alpha_k|^2 \hat{\psi}^{-1} = Tr(I_B)\hat{\psi}^\dagger \implies \hat{\psi}^{-1} = D_U \hat{\psi}^\dagger$$

$$\implies \hat{\psi} = \frac{1}{\sqrt{D_U}} I = \frac{1}{\sqrt{D_\psi}} I \implies |\psi\rangle \text{ is fully entangled.}$$

Literature

[1] Dan Stahlke and Robert B. Griffiths.
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[2] Akihito Soeda, Peter S. Turner, and Mio Murao.
Entanglement cost of implementing controlled-unitary operations.
arXiv:1008.1128, 2010.

[3] Robert B. Griffiths, Shengjun Wu, Li Yu, and Scott M. Cohen.
Atemporal diagrams for quantum circuits.
Phys. Rev. A, 73(5):052309, May 2006.

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